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Self-avoiding walks and polygons on non-Euclidean lattices

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Abstract. Self-avoiding walks and polygons on four non-Euclidean lattices are studied by the method of series expansions. It appears that the polygons and walks have different critical points from each other, and that both are in a different universality class to their Cartesian lattice counterparts. An exact solution is given for one lattice.

1. Introduction

It is well known that the spatial dimension of a physical system and the symmetries of the interactions are primary determinants of the critical behaviour of physical systems. Another characteristic that has been investigated for Ising systems [1] is the effect of the curvature of the space in which the system is embedded. In this paper we study the self-avoiding walk (SAW) problem in the Poincaré disk, which is a space of uniform negative curvature. In [1], two different lattices were studied, and for a detailed description of the lattices, their geodesics and symmetry groups, the reader is referred to the cited article. We have studied the SAW problem on the same lattices, as well as on two additional lattices. On one of these an exact solution is obtained.

The SAW problem has been extensively studied for many years by various methods [2, 3, 5–10]. Although it still remains unsolved in a mathematically rigorous fashion, there are some methods which yield significant and important results, a number of which are believed to be exact. Two numerical methods, exact enumeration and Monte Carlo are the most common methods to study these problems. The Monte Carlo method is based on a random sample of long SAW (10^2–10^7 steps) generated on a computer, and appropriately averaged. In the exact enumeration method, all SAW up to a certain length n are enumerated and later a variety of methods of series analysis such as the ratio method, the method of Padé approximants or the method of differential approximants are used to determine the asymptotic behaviour of the SAW generating function. Such approximations are based on both known properties of SAW and conjectures about SAW, which we now briefly review.

Let c_n denote the total number of n step SAW, u_n the total number of self-avoiding returns (SAR) and p_n the total number of self-avoiding polygons (SAP), on a given lattice. Since the SAR are just oriented, rooted polygons, we have

$$p_n = \frac{u_n}{2n}. \tag{1.1}$$

On regular lattices, the asymptotic form for large n is believed to be given by

$$c_n \sim A\mu_w^n n^g \tag{1.2}$$

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for SAW, and g is often denoted $\gamma - 1$, and

$$u_n \sim B\mu_p^n n^{-h} \tag{1.3}$$

for SAR where h is often denoted $2 - \alpha$†.

Here $f(n) \sim g(n)$ means that f is asymptotic to g as $n \to \infty$; i.e.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1. \tag{1.4}$$

For regular and Euclidean lattices

$$\mu_w = \mu_p = \mu \tag{1.5}$$

where μ is a constant associated with each lattice, known as the connective constant. The connective constant μ is not a universal constant, but depends on the underlying lattice. For the hexagonal lattice it is believed [18] that $\mu = (2 + \sqrt{(2)})^{\frac{1}{2}}$. The existence of μ was first proved rigorously by Hammersley and Morton [12], who later presented further rigorous results concerning SAW and SAP in [13–17].

Numerical estimates of g and h show their independence of the lattice structure in a given dimension. This lack of dependence is a manifestation of *universality*, and models with the same exponents are said to be in the same *universality class*. The values of g and h are as follows.

In two dimensions :

$$g = \tfrac{11}{32} \qquad h = \tfrac{3}{2}. \tag{1.6}$$

These two-dimensional results are believed to be exact [18].

In three dimensions numerical estimates are [19, 11]

$$g = 0.161(1) \qquad h = 1.7637(15). \tag{1.7}$$

In dimensions greater than four, g and h attain their mean-field values of 0 and $d/2$ respectively [2]. In four dimensions, mean-field values with logarithmic corrections are observed [2].

The motivation for the present investigation is to find out which of the above properties of SAW and SAR apply for some non-Euclidean lattices in the hyperbolic plane. In this work we enumerate series for SAW and SAR on four different non-Euclidean tessellations (hyperlattices for short) and attempt to estimate the connective constant μ, as well as the exponents g and h.

Two of these lattices (referred to in the following section as the $\{3, 7\}$ and $\{5, 5\}$ hyperlattices) were first considered in [20, 21] in connection with statistical mechanics, and later by Rietman *et al* [1], where the Ising model was investigated by the method of series expansions. (We mention in passing that these two lattices were denoted $(7, 3)$ and $(5, 5)$ in [1], but we have used the more conventional notation $\{p, q\}$, in which q p-gons meet at each vertex.)

The possibility of finding some new property which determines the critical behaviour of the system (such as the curvature of the space in which the system is placed) was the reason for the interest in this type of lattice. No such characterization has yet been identified. However, we find the new, and somewhat surprising result that the critical points for the SAW and SAP generating functions differ. (The analogous result for the Ising model has not been investigated, as the series for the specific heat of the Ising model—which is the analogue of the SAP generating function—was too short to be analysed [1]).

† For loose-packed lattices, $u_n = 0$ when n is odd. For simplicity we write (1.3) with the understanding that it holds only for $u_n \neq 0$.

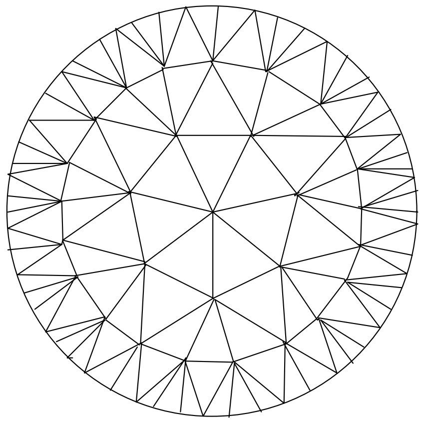


Figure 1. $\{3, 7\}$ lattice.

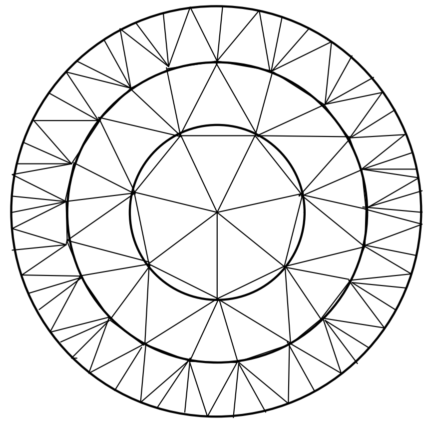


Figure 2. $\{3, 7\}$ lattice with layers.

The third lattice (denoted the $(6, 3)$ lattice) was consider by Lundt [23] where the free energy was evaluated (by solving a dimer problem) for an Ising model on this lattice. This lattice was constructed as a lattice which is homogeneous under a group other than translations, on which the dimer problem can be solved. Another viewpoint is that this lattice is embeddable in the $\{6, 4\}$ hyperbolic lattice. As we show in section 5, the SAW- and SAP-generating functions are exactly solvable on this lattice.

The fourth lattice, the so-called $T(2, 3, 7)$ hyperlattice, is an irregular hyperbolic lattice.

2. Description of the hyperlattices

We consider four different hyperlattices. The first two can be characterized by two integers $\{p, q\}$, where q is the number of neighbours of each vertex, and p is the number of sides of each face or polygon. The other two are related to the hyperlattices in a manner specified below. They are all effectively infinite dimensional lattices, in that they cannot be embedded in $\mathcal{R}^n$. While sharing this property with Cayley trees, they are more complex than Cayley trees in that they permit loops.

2.1. $\{3, 7\}$ *hyperlattice*

This is a regular hyperlattice, constructed on the hyperbolic plane using triangles with angles: $\frac{\pi}{7}$, $\frac{\pi}{7}$, $\frac{\pi}{7}$ (see figure 1). We can consider this lattice as a union of shells or layers, as shown in figure 2.

The number of sites n_k on the kth layer can be calculated recursively by

$$n_k = 3n_{k-1} - n_{k-2} \tag{2.1}$$

where $n_0 = 7$, $n_1 = 21$.

The generating function for the number of sites per layer defined by

$$P(x) = x^{-2} \sum_{k=2}^{\infty} n_k x^k \tag{2.2}$$

satisfies the algebraic equation

$$P(x) = 3n_1 + 3x P(x) - n_0 - n_1 x - P(x) x^2 \tag{2.3}$$

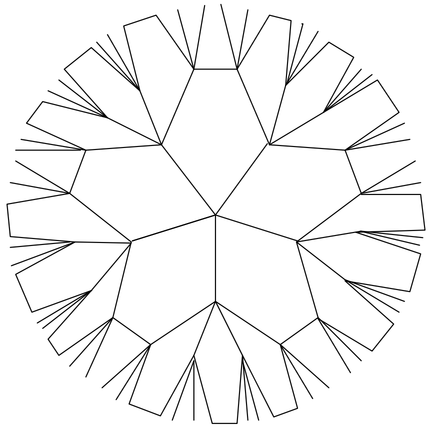


Figure 3. {5, 5} lattice.

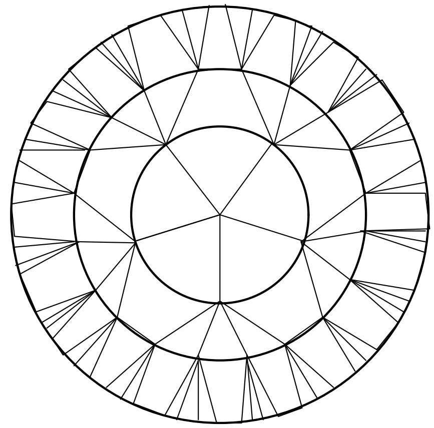


Figure 4. {5, 5} lattice with layers.

which may be solved to yield

$$P(x) = \frac{56 - 21x}{1 - 3x + x^2}. \tag{2.4}$$

The number of sites on each shell is equal to the appropriate coefficient in the Taylor expansion:

$$\begin{aligned} P(x) = 56 &+ 147x + 385x^2 + 1008x^3 + 2639x^4 + 6909x^5 + 18\,088x^6 + 47\,355x^7 \\ &+ 123\,977x^8 + 324\,576x^9 + 849\,751x^{10} + 2\,224\,677x^{11} + 5\,824\,280x^{12} \\ &+ 15\,248\,163x^{13} + 39\,920\,209x^{14} + 104\,512\,464x^{15} + 273\,617\,183x^{16} \\ &+ 716\,339\,085x^{17} + 1\,875\,400\,072x^{18} + 4\,909\,861\,131x^{19} \\ &+ 12\,854\,183\,321x^{20} + \mathrm{O}(x^{21}). \end{aligned} \tag{2.5}$$

The general solution of the recursion (2.1) is

$$n_k = \frac{7}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{k+1}\right). \tag{2.6}$$

A more detailed description of the construction of this lattice is given in [1].

2.2. {5, 5} *hyperlattice*

This is a regular hyperlattice, constructed on the hyperbolic plane by pentagons (see figure 3). We can also consider this lattice as a union of layers, as shown in figure 4.

The number of sites n_k on the kth layer can also be calculated recursively [22]:

$$n_k = 3n_{k-1} + n_{k-2} + 3n_{k-3} - n_{k-4} \tag{2.7}$$

where $n_0 = 5$, $n_1 = 20$, $n_2 = 70$, $n_3 = 245$.

The generating function defined by

$$P(x) = x^{-4}\sum_{k=4}^{\infty} n_k x^k \tag{2.8}$$

satisfies the algebraic equation

$$P(x) = \frac{860 + 435x + 665x^2 - 245x^3}{1 - 3x - x^2 - 3x^3 + x^4}. \tag{2.9}$$

Hence $n_k \sim a^k$ where $a = \frac{3+b+\sqrt{14+6b}}{4}$ and $b = \sqrt{21}$. The number of sites on each shell is equal to the appropriate coefficient in the Taylor expansion:

$$\begin{aligned} P(x) = 860 &+ 3015x + 10\,570x^2 + 37\,060x^3 + 129\,935x^4 + 455\,560x^5 + 1\,597\,225x^6 \\ &+5\,599\,980x^7 + 19\,633\,910x^8 + 68\,837\,825x^9 + 7\,241\,350\,100x^{10} \\ &+846\,189\,875x^{11} + 2\,966\,799\,290x^{12} + 10\,401\,800\,220x^{13} \\ &+36\,469\,419\,475x^{14} + 127\,864\,266\,640x^{15} + 448\,300\,820\,765x^{16} \\ &+1\,571\,773\,187\,140x^{17} + 5\,510\,743\,762\,630x^{18} + 19\,321\,042\,670\,685x^{19} \\ &+67\,740\,890\,515\,340x^{20} + \mathrm{O}(x^{21}). \end{aligned} \tag{2.10}$$

A more detailed description of the construction of this lattice is given in [1]

2.3. (6, 3) *hyperlattice*

This is a regular hyperlattice which consists of hexagons, each one connected to three others along alternating faces in such a way as to form an infinite tree (see figure 5). It is also a lattice embeddable in the {6, 4} hyperbolic lattice. We can also consider this lattice as a union of layers as demonstrated in figure 6.

It is easy to find the number of sites on each layer: $n_0 = 1$, $n_1 = 2$, $n_3 = 2 * 4$, $n_4 = 2 \times 4$, $n_5 = 2 \times 4$, $n_6 = 2 \times 4^2$, $n_7 = 2 \times 4^2$, $n_8 = 2 \times 4^2$, $n_9 = 2 \times 4^3$, $n_{10} = 2 \times 4^3$, ... The construction of this lattice has been described in more detail by Lundt [23].

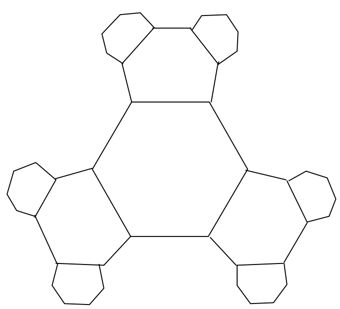


Figure 5. (6, 3) lattice.

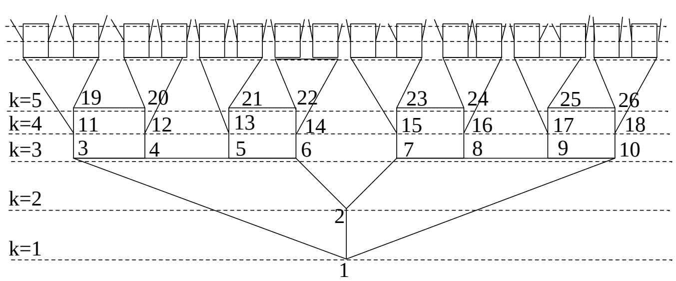


Figure 6. (6, 3) lattice with layers.

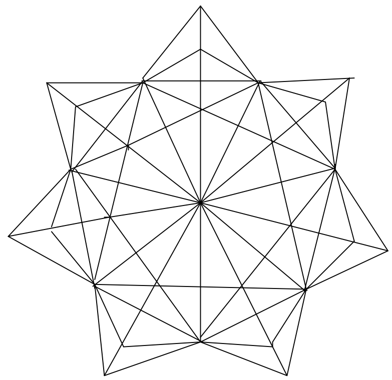


Figure 7. T(2, 3, 7) lattice.

2.4. T(2, 3, 7) hyperlattice

This is an irregular hyperlattice constructed on the hyperbolic plane using triangles with angles: $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$ (see figure 7). It can also be viewed as a triangulation of the regular {8, 4} hyperlattice, in which each octagon is triangulated by 16 triangles, with a common vertex at the centre of the octagon. The construction of this lattice is described in [24].

3. Enumeration methods

To enumerate SAW and SAR on hyperlattices, the back-tracking method was used (similar to that discussed by Grassberger [25]). We encountered the usual enumeration problems associated with problems of exponential complexity. First, the numbers involved get large very quickly (in fact exponentially), so that the time needed for the enumeration of these numbers grows exponentially as well. Secondly, for direct enumeration on such lattices, memory is also a problem. When moving from layer to layer the number of sites which have to be stored also increases exponentially.

To illustrate these difficulties, consider the {3, 7} lattice. If we want to produce a nearest-neighbour look-up table for SAW of lengths 18 steps, we need a table which contains 169 104 712 sites (each site has seven neighbours). This number is even bigger for the {5, 5} lattice. To avoid this problem we tried two methods.

In the first method, instead of producing a look-up table we found nearest-neighbours as needed. First, we observed that we have only two types of connections between layers for a {3, 7} lattice, and four types for a {5, 5} lattice as shown in figures 8 and 9, and later we observed a pattern which allows us to construct these two lattices.

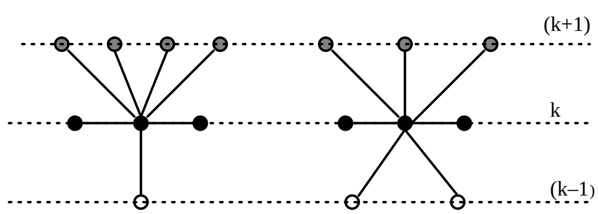


Figure 8. There are two types of connections between layers on the {3, 7} lattice.

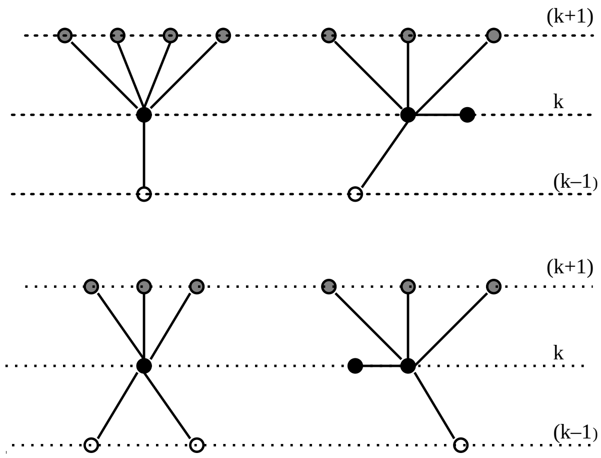


Figure 9. There are four types of connections between layers on the {5, 5} lattice.

For the {3, 7} lattice the following rule was used. On each layer we chose a local origin. The local origin on the kth layer $O(k)$ is a site connected to the local origin of the previous layer $O(k-1)$ in such way that its left-hand neighbour on the same layer is not connected with the local origin on the $(k-1)$th layer. We assign labels to each site on the kth layer starting from the local origin $O(k)$ with 1, and increasing in a clockwise manner incrementing by one at each step (see figure 10). $O(1)$ is any point on the first layer.

Let us denote the nth point on the kth layer by (k, n). Hence the two nearest neighbours of (k, n) on the same layer are $(k, n-1)$ and $(k, n+1)$ (with appropriate adjustment when the right-hand neighbour is the local origin).

To find the nearest neighbours on the previous and next layer we notice that starting from the second layer we have a regular pattern of connections between sites (see figure 10). Any three consecutive sites on one layer are connected to nine consecutive sites on the next layer; one site is connected with three sites on the next layer, and two of them with four. We

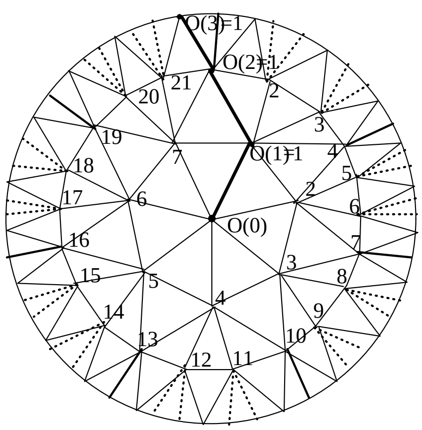


Figure 10. {3, 7} lattice with labelled vertices.

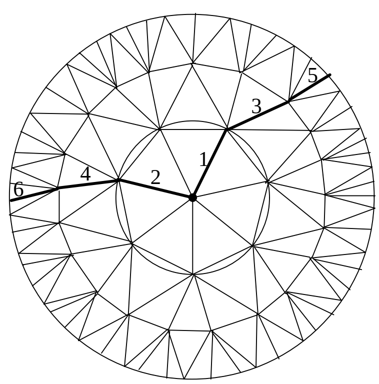


Figure 11. Odd–even walk.

wrote a computer program to utilize this pattern when moving between layers, and hence finding nearest neighbours for any given site. The same method was implemented for the $\{5, 5\}$ lattice.

The second method, suggested by Oitmaa [22], is based on the odd–even walks. Without overlapping, the first two steps of the walk both start from the origin in the zeroth layer. We consider these two steps as the first steps in two different paths. Thereafter, each odd step is attached to the path formed by the first step, and each even step is attached to the second path formed by the second step. The paths cannot meet (see figure 11).

The enumeration of all such walks for all possible combinations of the first two steps gives the total number of all SAW for open walks, and SAR for closed walks. In this way we significantly reduced the size of the look-up table, as for $(2n)$-step walks we need to remember only the first n layers. To produce the look-up table we used a program which was written by Oitmaa. The total number of sites for layers (0–10) for the $\{5, 5\}$ lattice is 637 341 and 76 615 for the $\{3, 7\}$ lattice. Memory is therefore not a problem for walks up to about 20 steps.

The second method turns out to be faster than the first, and we were able to enumerate the number of SAW and SAR up to 17 steps for the $\{3, 7\}$ lattice which took almost 800 h (on a relatively slow IBM RISC 6000/530) and up to 19 steps for the $\{5, 5\}$ lattice which took 43 h.

For the $(6, 3)$ lattice we used the first method. It was not necessary to produce the look-up table as it is not very difficult to find the nearest neighbours using the representation of the lattice given in figure 6. We were able to enumerate the number of SAW and SAR up to 23 steps, which took almost 6 h of computer time. However, as we subsequently show, it is possible to solve for this lattice analytically.

We cannot use the even–odd walk method for $\mathrm{T}(2, 3, 7)$ because this hyperlattice is irregular. Since the number of sites on each layer grows less rapidly than that for the $\{3, 7\}$ or the $\{5, 5\}$ lattices, we could easily produce the lookup table moving from layer to layer (the total number of sites (0–16) is equal to 53 214). It took almost 16 h to enumerate the number of SAW and SAR up to 15 steps. All the results are given in table 1.

4. Analysis of series

To analyse the series we used both the method of differential approximants, and variants of the ratio method. Both methods are described in detail in [4].

4.1. Summary of the method

In the method of differential approximants the function f is represented by the Kth order differential equation

$$\sum_{i=0}^{K} Q_i(z)\mathcal{D}^i f(z) = P(z) \tag{4.1}$$

where $\mathcal{D} = z\frac{\mathrm{d}}{\mathrm{d}z}$ and Q_i and P are polynomials in z of degree N_i and M, respectively. $Q_{K,0} = 1$ is set to 1, and in the homogeneous case $P(z) = 0$, $Q_{0,0} = 0$, which forces the point at the origin to be a regular singular point. The symbol $[M/N_0; N_1; \ldots; N_K]$ denotes the approximant. For $K = 1$ the approximant $[M/N_0; N_1]$ is appropriate to represent

Table 1. Number of SAW and SAR on four hyperlattices.

	SAW{3, 7}	SAR{3, 7}	SAW{5, 5}	SAR{5, 5}	SAW(6, 3)	SAR(6, 3)	SAWT(2, 3, 7)	SART(2, 3, 7)
1	7	—	5	—	3	—	14	—
2	42	—	20	—	6	—	56	—
3	238	14	80	—	12	—	378	28
4	1330	28	320	—	24	—	1848	70
5	7392	70	1270	10	48	—	10 332	196
6	40 978	196	5050	—	92	4	53 438	700
7	226 842	602	20 080	—	180	—	283 752	2604
8	1 254 666	1960	79 800	40	352	—	1 480 836	9436
9	6 936 272	6510	317 190	—	688	—	7 761 544	35 280
10	38 335 752	22 050	1 260 760	—	1334	10	40 502 448	134 288
11	211 840 650	76 076	501 100	220	2600	—	211 423 772	519 344
12	1 170 496 572	265 944	19 917 020	—	5064	—	1 102 131 170	2 039 170
13	6 467 010 298	939 120	79 163 280	—	9864	—	5 743 708 712	8 109 304
14	35 728 910 462	3 346 630	314 645 160	1400	19 180	28	29 916 531 610	32 669 672
15	197 389 980 690	12 019 560	1 250 602 250	30	37 340	—	155 779 504 398	133 484 064
16	1 090 494 503 392	43 456 448	4 970 696 950	—	72 680	—		552 469 624
17	6 024 449 653 376	158 036 760	19 756 733 220	9520	141 472	—		
18		577 723 020	78 525 928 980	540	275 268	84		
19			312 112 399 000	—	535 752	—		
20				67 640	1 042 680	—		
21					2 029 280	—		
22					3 949 064	264		
23					7 685 552	—		
24					14 957 224	—		

functions with a single non-analytic singularity

$$f(z) \sim A(z) + B(z)\left(1 - \frac{z}{z_i}\right)^{-\gamma} \tag{4.2}$$

and for $K > 1$ functions with more than one singularity,

$$f(z) \sim A(z) + B(z)\left(1 - \frac{z}{z_i}\right)^{-\gamma} \times\left(1 + C_1(z)\left(1 - \frac{z}{z_c}\right)^{\delta_1} + \cdots + C_{K-1}(z)\left(1 - \frac{z}{z_c}\right)^{\delta_{K-1}}\right) \tag{4.3}$$

which may be confluent, as shown in (4.3) or non-confluent. Usually the first- and second-order linear inhomogeneous differential approximants are constructed, with inhomogeneous polynomials of degree 1–8. This can be done by matching the series coefficients to a Kth order differential equation. For a fixed number of terms, critical point and critical exponent are estimated. Later, to summarize the data, means of estimates of the critical point and exponent are taken for a fixed value of the order of the coefficients used in the construction of the approximants (this order is denoted by n). For each value of n there are, say, L non-defective approximants with an error, which is taken as twice the standard deviation. Final estimates of the critical point and exponent are obtained by taking all estimates with an error not bigger than $4*$(minimum error) with a weight which depends on the error. For example, for the entries for the critical exponent $\gamma_i \pm \epsilon_i$ $(i = 1, \ldots, M)$ estimate of γ can be found as

$$\langle\gamma\rangle = \sum_{i=1}^{M} \frac{(\gamma_i)}{\epsilon_i}\left(\sum_{i=1}^{M} \frac{1}{\epsilon_i}\right)^{-1} \tag{4.4}$$

with an error given by

$$\langle\epsilon\rangle = \sqrt{M}\left(\sum_{i=1}^{M} \frac{1}{\epsilon_i}\right)^{-1}. \tag{4.5}$$

If the value of the critical point is known, or a good estimate can be found, a biased approximant can be constructed, forcing the approximant to be singular at the given critical point.

To calculate the differential approximant we used the Fortran program NEWGRQD. To analyse the unbiased and biased approximant we used another Fortran program called TABUL. These programs and their description can be found in [4].

4.2. Results of analysis

4.2.1. {3, 7} *lattice.*

SAW. Results of the analysis of the SAW-generating function are shown in table 2. As explained above, each row gives the mean and twice the standard deviation of the the critical point and critical exponent estimates. The *estimates* were obtained using the procedure described before (equations (4.4), (4.5)). Combining the first- and second-order differential approximants gives

$$\frac{1}{\mu} = 0.181\,015(15) \qquad g = 0.000(3). \tag{4.6}$$

Table 2. SAW on {3, 7} lattice. Summary of critical point and exponent estimates from first- and second-order differential approximants.

n	Critical point		Critical exponent		L
$K = 1$					
8	0.181 3466	0.000 1290	−1.011 2081	0.005 1242	4
9	0.181 1831	0.000 1678	−1.005 7662	0.004 9784	6
10	0.181 0169	0.000 0584	−1.000 5457	0.001 8296	7
11	0.181 0165	0.000 0449	−1.000 2111	0.000 9236	10
12	0.181 0192	0.000 0117	−1.000 0896	0.000 5353	11
13	0.181 0214	0.000 0035	−1.000 3190	0.000 2065	11
14	0.181 0189	0.000 0018	−1.000 2060	0.000 1286	9
15	0.181 0170	0.000 0070	−0.999 6348	0.002 0328	6
16	0.181 0154	0.000 0012	−0.999 9921	0.000 0799	5
Estimates	0.181 0176	0.000 0010	−1.001 6252	0.000 0698	
$K = 2$					
13	0.180 9847	0.000 1158	−0.995 1051	0.012 1457	5
14	0.181 0013	0.000 0296	−0.997 9865	0.003 7320	5
15	0.181 0157	0.000 0200	−0.999 5591	0.000 9748	5
16	0.181 0139	0.000 0085	−0.999 6438	0.001 0029	8
Estimates	0.181 0123	0.000 0086	−0.999 4120	0.000 7560	

Table 3. SAR on {3, 7} lattice. Summary of critical point and exponent estimates from first- and second-order differential approximants.

n	Critical point		Critical exponent		L
$K = 1$					
11	0.251 9716	0.002 3757	0.410 7290	0.086 4030	6
12	0.251 8882	0.003 6432	0.461 0302	0.487 2579	10
13	0.251 3263	.002 0893	0.541 1557	0.319 7514	4
14	0.251 1697	0.001 4521	0.535 2664	0.226 3169	7
15	0.250 5312	0.001 3681	0.637 6466	0.282 6303	7
Estimates	0.251 2248	0.000 8621	0.488 4528	0.088 2695	
$K = 2$					
11	0.245 1500	0.006 3694	1.144 7361	0.251 7024	$2x$
12	0.255 3495	0.000 1956	0.053 0493	0.016 1568	$2x$
13	0.251 1197	0.006 2944	0.605 4072	0.864 7347	5
14	0.251 0471	0.002 1844	0.503 9439	0.487 0687	5
15	0.250 5469	0.001 6094	0.629 3797	0.333 8532	5
Estimates	0.250 8054	0.001 3991	0.583 4069	0.279 1446	

SAR. Results of the analysis of the SAR generating function are shown in table 3. Combining the first- and second-order differential approximants results gives

$$\frac{1}{\mu} = 0.250(3). \tag{4.7}$$

Table 4. SAW on {5, 5} lattice. Summary of critical point and exponent estimates from first- and second-order differential approximants.

n	Critical point		Critical exponent		L
$K = 1$					
11	0.251 5943	0.000 0034	−1.000 0023	0.000 0446	10
12	0.251 5958	0.000 0014	−1.000 0178	0.000 0489	9
13	0.251 5952	0.000 0022	−1.000 0804	0.000 4344	11
14	0.251 5950	0.000 0004	−1.000 0001	0.000 0051	10
15	0.251 5951	0.000 0001	−0.999 9996	0.000 0030	8
16	0.251 5951	0.000 0001	−1.000 0043	.000 0239	7
17	0.251 5950	0.000 0000	−1.000 0000	.000 0001	5
Estimates	0.251 5950	0.000 0000	−1.000 0000	0.000 0001	
$K = 2$					
13	0.251 5931	0.000 0030	−0.999 8584	0.000 2672	5
14	0.251 5944	0.000 0012	−0.999 9590	0.000 0990	7
15	0.251 5950	0.000 0026	−0.999 9489	0.000 2865	7
16	0.251 5951	0.000 0008	−0.999 9961	0.000 0407	8
17	0.251 5950	0.000 0001	−0.999 9948	0.000 0112	7
Estimates	0.251 5950	0.000 0001	−0.999 9951	0.000 0124	

A ratio analysis allows us to give a similar estimate for μ, and in addition the assumption that $\mu = 4$ exactly gives the exponent estimate $h = 1.50(10)$. Thus it is clear that on the {3, 7} lattice SAW and SAR have different connective constants.

4.2.2. {5, 5} *lattice.*

SAW. Results of the analysis of the SAW-generating function are shown in table 4. We estimate that

$$\frac{1}{\mu} = 0.251\,5950(1) \qquad g = 0.0. \tag{4.8}$$

SAR. The series was too short to estimate the connective constant and exponent for SAR on the {5, 5} lattice.

4.2.3. (6, 3) *lattice.*

SAW. Results of the analysis of the SAW-generating function are shown in table 5. Combining the first- and second-order differential approximants gives

$$\frac{1}{\mu} = 0.513\,84(2) \qquad g = 0.000(1). \tag{4.9}$$

SAR. Again, the series is too short to enable us to estimate the connective constant and exponent for SAR, but in the next section we give the exact solution for this lattice, from which we see that $\frac{1}{\mu} = 0.707\,106\ldots$ and $h = 1.5$.

Table 5. SAW on (6, 3) lattice. Summary of critical point and exponent estimates from first- and second-order differential approximants.

$K = 1$					
n	Critical point		Critical exponent		L
12	0.512 7440	0.003 4333	−1.004 1410	0.028 3589	4
13	0.514 4424	0.000 7715	−1.012 0279	0.014 4167	5
14	0.514 3630	0.001 1776	−1.014 6700	0.043 7898	7
15	0.514 1398	0.000 8242	−1.008 4590	0.028 9819	5
16	0.513 8722	0.000 2722	−1.001 4012	0.006 0802	9
17	0.513 8864	0.000 0571	−1.000 8782	0.001 1437	11
18	0.513 8892	0.000 1814	−1.001 2543	0.006 7765	13
19	0.513 8549	0.000 0491	−1.000 3415	0.001 3070	13
20	0.513 8411	0.000 0164	−1.000 0248	0.000 3386	15
21	0.513 8483	0.000 0143	−1.000 1057	0.000 7745	8
22	0.513 8451	0.000 0059	−1.000 2026	0.000 2616	14
23	0.513 8415	0.000 0030	−1.000 0612	0.000 1551	10
Estimates	0.513 8427	0.000 0028	−1.000 0940	0.000 1310	
$K = 2$					
18	0.513 8042	0.000 8158	−0.991 6375	0.024 1171	5
19	0.513 8877	0.000 6843	−0.997 5011	0.018 8578	5
20	0.513 7973	0.000 0793	−0.997 8040	0.005 3063	6
21	0.513 8324	0.000 0942	−0.999 4792	0.006 0643	4
22	0.513 8103	0.000 2040	−0.992 3375	0.039 8659	9
23	0.513 8226	0.000 0132	−0.998 4988	0.002 1183	8
Estimates	0.513 8226	0.000 0132	−0.998 5360	0.002 0984	

4.2.4. $T(2, 3, 7)$ lattice.

SAW. Results of the analysis of the SAW-generating function are shown in table 6. We estimate that

$$\frac{1}{\mu} = 0.1924(2) \qquad g = 0.00(2). \tag{4.10}$$

SAR. Although the polygon series was the same length as that for SAW we found that the series was rather badly behaved. We were unable to make any estimate of the critical point or exponent by the method of differential approximants. From the ratio method it seems possible to estimate the critical point as $1/\mu = 0.2$ but with a rather large uncertainty.

5. Exact solutions

For the (6, 3) lattice the number of SAP and SAW can also be evaluated exactly. Let f_1 be the generating function for walks on the (6, 3) lattice which start at ●, move first in the $+x$ direction and return to the adjacent south-west site.

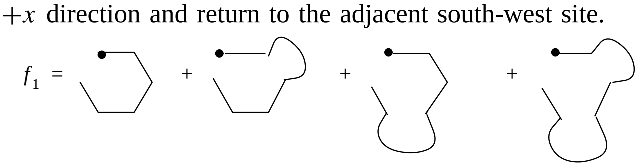


Table 6. SAW on T(2, 3, 7) lattice. Summary of critical point and exponent estimates from first- and second-order differential approximants.

n	Critical point		Critical exponent		L
$K = 1$					
8	0.192 2625	0.005 5139	−1.073 7759	0.034 5037	5
9	0.193 6776	0.002 1089	−1.102 9940	0.184 9718	7
10	0.192 5311	0.001 1313	−1.029 8179	0.039 3297	7
11	0.192 6558	0.000 5885	−1.032 0224	0.043 6121	10
12	0.192 5282	0.000 4061	−1.025 3481	0.017 9328	10
13	0.192 5410	0.000 0918	−1.021 0954	0.009 3643	9
14	0.192 5028	0.000 1319	−1.018 7711	0.018 2577	10
Estimates	0.192 5253	0.000 0766	−1.027 7402	0.008 1200	
$K = 2$					
10	0.191 9073	0.001 7501	−0.985 1313	0.023 8791	$3x$
11	0.192 5378	0.002 6857	−0.951 1558	0.359 8304	4
12	0.192 4511	0.000 2711	−1.012 0410	0.031 6081	5
13	0.192 5054	0.000 5925	−1.005 2997	0.055 3710	6
14	0.192 4704	0.000 1033	−1.011 2683	0.018 4425	5
Estimates	0.192 4651	0.000 1058	−1.010 4663	0.016 6671	

There are four possible configurations, as shown in the above figure, and by inspection one can write down an algebraic equation satisfied by the generating function, namely

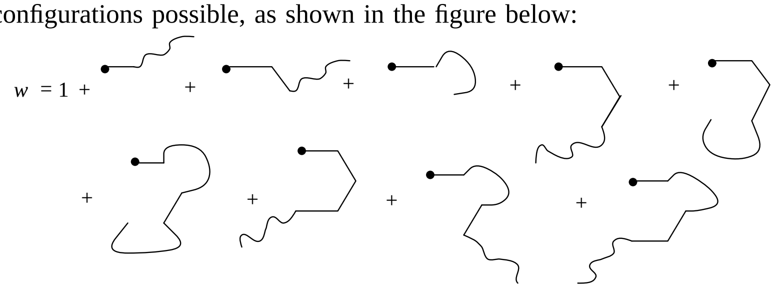
$$f_1 = x^5 + x^4 f_1 + x^4 f_1 + x^3 f_1^2. \tag{5.1}$$

Hence, taking the negative square-root in the solution of the above quadratic in order to match the boundary condition that the first term in f_1 is x^5, gives

$$f_1 = \frac{1 - 2x^4 - \sqrt{1 - 4x^4}}{2x^3} = x^5 + 2x^9 + 5x^{13} + 14x^{17} + 42x^{21} + 132x^{25} + \cdots. \tag{5.2}$$

These coefficients are immediately recognizable as Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, $n \geqslant 1$. Thus we find $\mu_p = 1/\sqrt{2}$ and $h = \frac{3}{2}$, in agreement with both the exponent found numerically for SAP on the {3,7} lattice, and the exact value found on regular lattices!

The generating function for SAW can also be obtained with somewhat greater difficulty. Let w be the generating function for walks on the (6, 3) lattice which start at ●, move first in the $+x$ direction, and do not end on an adjacent site. There are nine such distinct configurations possible, as shown in the figure below:



By inspection, we can write the algebraic equation for the generating function as follows:

$$w = 1 + xw + x^2 w + x f_1 + x^3 w + x^3 f_1 + x^2 f_1^2$$

$$+x^4w + x^2 f_1 w + x^3 f_1 w. \tag{5.3}$$

Then the generating function for the total number of walks is obtained by construction from this generating function and the polygon-generating function obtained above, and is given by:

$$2w(1 + f_1) - 1 + x(2w - 1) = 2w(1 - f_1 + x) - 1 - x. \tag{5.4}$$

The generating function for SAW $C(x)$ can be written in the form

$$C(x) = \frac{A(x) + B(x)\sqrt{1 - 4x^4}}{x^6 D(x)} \tag{5.5}$$

where

$$A(x) = 1 - x - x^3 - 6x^4 + 2x^5 - 3x^6 + 5x^8 - 3x^9 - x^{10} - 2x^{11} - 2x^{12}$$
$$B(x) = -1 + x + x^3 + 4x^4 + 2x^6 - 2x^8$$
$$D(x) = -1 + x + x^3 + 4x^4 + 2x^5.$$

The polynomial $D(x)$ has one real zero at $x = 0.513\,839\,377\,437\,877\,403\,29$, which agrees with our numerical result, and verifies that $g = 0$, corresponding to a simple pole. These results also show explicitly that the connective constant for walks is different to that for polygons. Comparing (5.2) with (5.5), one sees that the singularity comes from the numerator in (5.2) and from the denominator in (5.5).

6. Conclusion

For all four lattices the series for SAW were found to behave in the manner conjectured by equation (1.2), with exponent $g = 0$. This is the same as the exponent for SAW on a Bethe lattice, and for the susceptibility of the Ising model on these lattices. It reflects the infinite-dimensional nature of the lattices. For the polygon series we were able to accurately estimate the connective constant on the {3, 7} lattice only, and we found that:

$$\mu_w \neq \mu_p. \tag{6.1}$$

This inequality was confirmed for the $(6, 3)$ lattice.

An anonymous referee has pointed out that the argument due to Hammersley (for a most accessible account, see [3]) for the existence of a critical point for SAW immediately extends to lattices of the type considered here. Hammersley subsequently proves the existence of the corresponding limit for SAP, and, further, shows that the two limits are equal. The referee raises the question as to why the Hammersley construction fails in the present case. Hammersley's proof involves concatenation of polygons, and bond deletions. This concept needs to be generalized to lattices in the hyperbolic plane. This would appear to be possible (though we have not done so), which would then allow the existence of the connective constant for SAP to be established. However, the proof of equality breaks down in two ways. First, the construction of polygons from four distinct walks makes use of symmetry properties of the Euclidean plane, which do not apply to the lattices under consideration, and secondly, the vital 'unfolding' transformation, whereby a class of unfolded SAW possesses the same connective constant as the class of SAW from which it was generated, also fails to hold.

Our results for a {3, 7} and {5, 5} lattice are consistent with the results found in [1] for the Ising model where for the {3, 7} lattice the critical point was found to be 0.1848 with exponent $g = 0.0$, and for the {5, 5} lattice the critical point was found to be 0.2520 again with exponent $g = 0.0$. (The Ising model study did not give estimates of the specific-heat critical point, which is the analogue of our SAP critical point.) Our values of the reciprocal of the connective constants are lower then the values of critical points obtained for the Ising model on the same lattice, which is not surprising. For the square lattice Ising model, the critical point is equal to $0.4142\ldots$, and the reciprocal of the connective constant obtained for SAW is equal to $0.379\,05\ldots$ [26].

It is interesting that, for both the SAW and the Ising model susceptibility, the generating function has a simple pole singularity at the critical point for these lattices. This suggests that the solution may be simpler for such lattices than for regular Euclidean lattices, a result confirmed by the exact solution given for the $(6, 3)$ lattice. However, that lattice has a tree-like dual structure, which accounts for the solvability of models on that particular lattice. The other lattices studied are less simply connected, and thus any exact solutions are likely to be more elusive.

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